KoAT: Automatic Complexity and Termination Analysis of Integer Programs

Nils Lommen, Éléanore Meyer, and Jürgen Giesl

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Goal: Infer (upper) runtime bounds and prove termination for integer programs

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  - Outer loop: $\mathcal{O}(x)$

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  - open-source complexity and termination analysis tool

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  [x]   [x - 1]
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  end
end
```

Bounds: (Ω(1), O(log(n)*n))

Took less than 1 Second

This number is the total time the Docker container was running. The overhead is about 1 second in most cases.

Show result in a new tab

Preprocessing

Found invariant $1+2\cdot X_2 \leq 3\cdot X_1 \wedge 1 \leq X_2$ for location l2

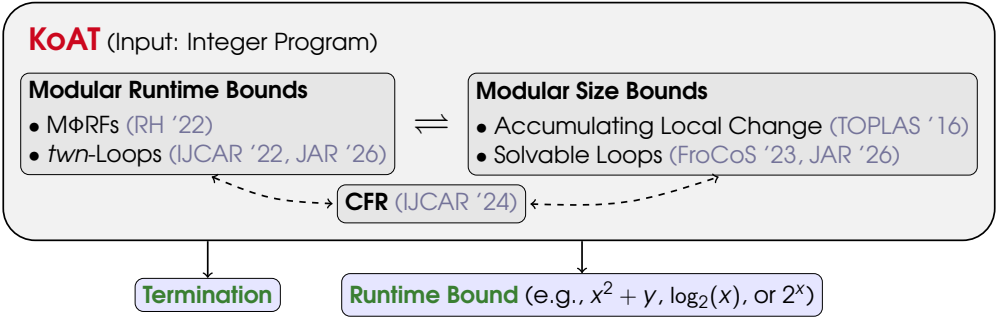
Problem after Preprocessing

Start: l0
Program_Vars: X_0, X_1, X_2
Temp_Vars:
Locations: l0, l1, l2
Return Locations:
Transitions:
t_0: $l0(X_0, X_1, X_2) \to l1(X_0, X_1, X_2)$
t_1: $l1(X_0, X_1, X_2) \to l2(X_0\text{-}1, X_0, 1)$:|: $1 \leq X_0$
t_3: $l2(X_0, X_1, X_2) \to l1(X_0\text{-}1, X_0, 1)$:|: $1+2\cdot X_2 \leq 3\cdot X_1 \wedge 1 \leq X_2$
t_2: $l2(X_0, X_1, X_2) \to l2(X_0, 2\cdot X_1, 3\cdot X_2)$:|: $1 \leq X_1 \wedge 1+X_2 \leq X_1 \wedge 1+2\cdot X_2 \leq 3\cdot X_1 \wedge 1 \leq X_2$

Show Graph | Show Dependency Graph

MPRF for transition t_1: $l1(X_0, X_1, X_2) \to l2(X_0\text{-}1, X_0, 1)$:|: $1 \leq X_0$ of depth 1:

Overview: KoAT



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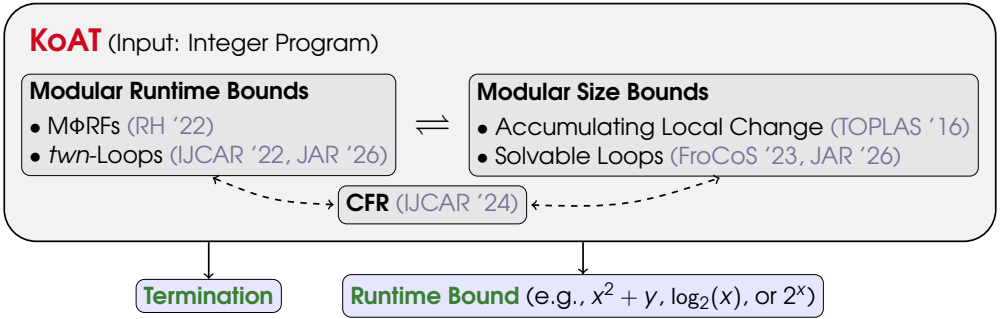

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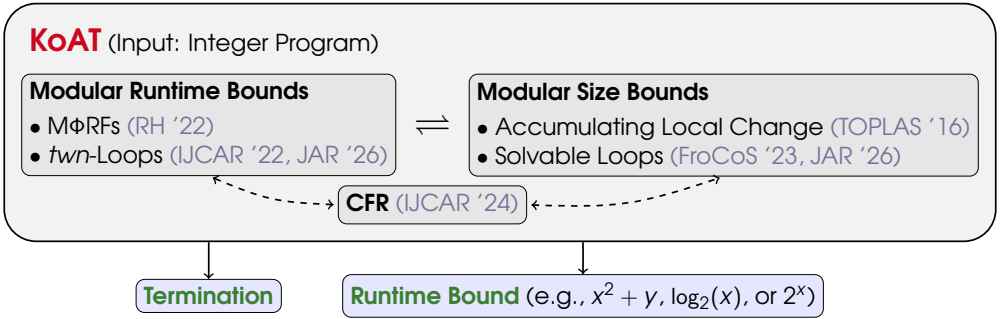

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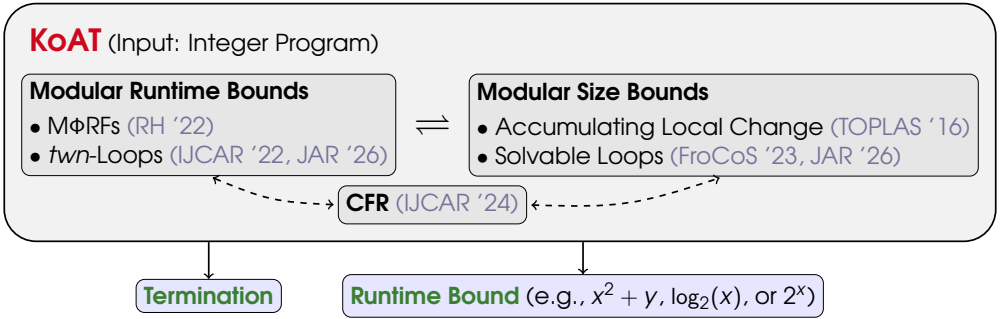

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  - Fallback technique to unroll/split loops and sort out paths.

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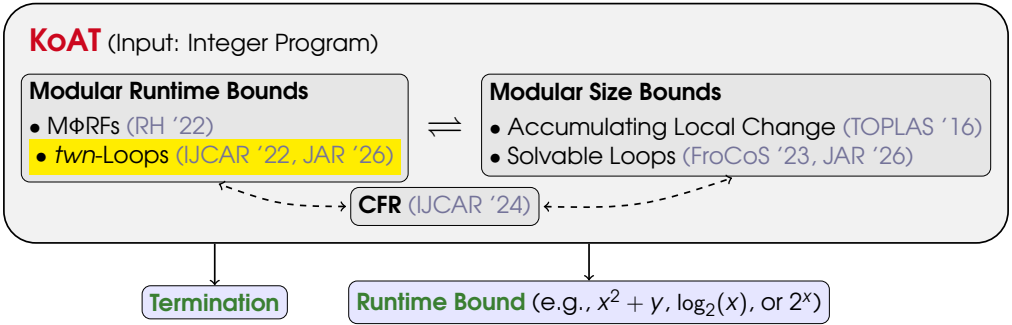

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Analyzing Triangular Weakly Non-Linear Loops

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while (y1 > y2 ∧ y2 > 0) do
    [y1]      [2 · y1]
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- Does the loop terminate?

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- Reduce termination to an existential formula over $\mathbb{Z}$.
- For terminating twn-loops always polynomial runtime bounds.
- KoAT infers the local runtime bound: $\log_{3/2}(y_1) + 1$

Lifting Bounds via Modularity

How do we get the global runtime of the nested program?

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while (x > 0) do
  [x, y1, y2] ← [x − 1, x, 1]
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- Inner loop has local bound $\log_{3/2}(y_1) + 1$. How often is it executed in total?

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- **Step 3 (Composition):** Substitute size bound into local runtime bound and multiply by outer runtime:

$$\mathcal{RB}_{\text{global}} \leq x \cdot (\log_{3/2}(x) + 1)$$

Live Demo

```
(GOAL COMPLEXITY)
(STARTTERM (FUNCTIONSYMBOLS l0))
(VAR x1 y1 y2)
(RULES
  l0(x1,y1,y2) -> l1(x1,y1,y2)
  l1(x1,y1,y2) -> l2(x1-1,x1,1) :|: x1 > 0
  l2(x1,y1,y2) -> l2(x1,2*y1,3*y2) :|: y1 > y2 && y1 > 0
  l2(x1,y1,y2) -> l1(x1-1,x1,1)
)
```

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Evaluation

Complexity Analysis (Complexity_ITS, 838 Benchmarks, < 647 Terminating)

	$\mathcal{O}(1)$	$\mathcal{O}(\log(n))$	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$< \omega$	$\mathrm{AVG}^{+}(\mathrm{s})$
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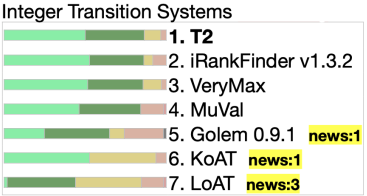

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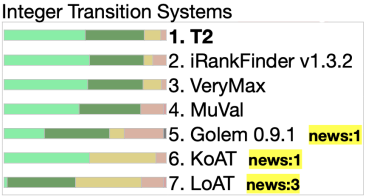
- KoAT + LoAT can solve 1157 benchmarks



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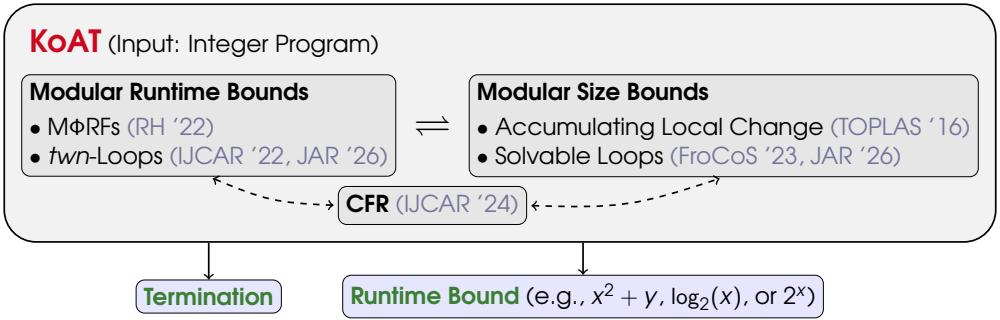

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- Probabilistic programs (TACAS '21, IJCAR '24)

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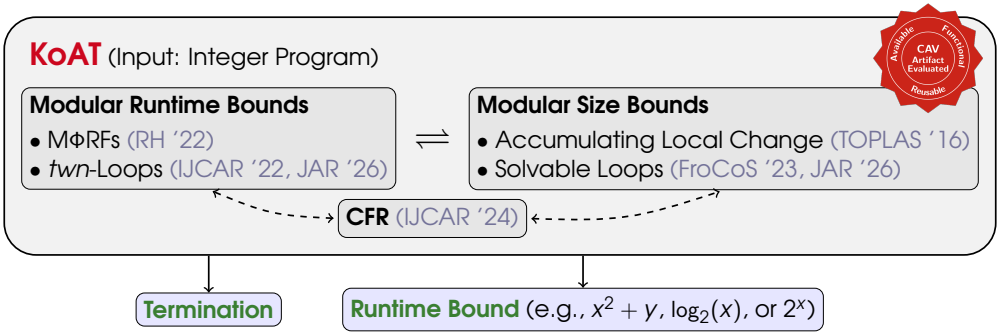

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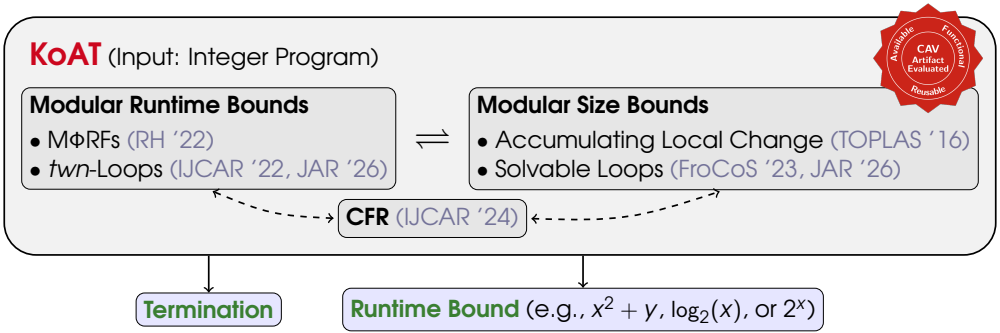

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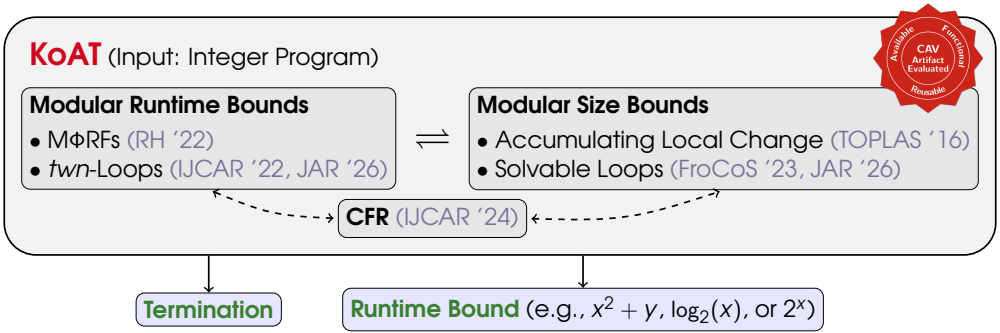

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