Modular Automatic Complexity Analysis of Recursive Integer Programs

Nils Lommen and Jürgen Giesl

Motivation

Goal: Infer (upper) runtime bounds for "real-world" programs

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Contribution: Extend modular complexity analysis approach by function calls

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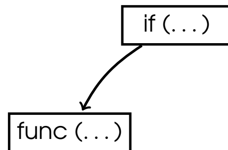

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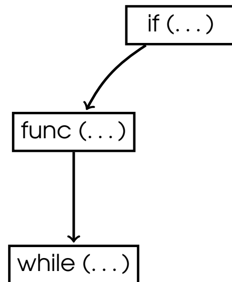

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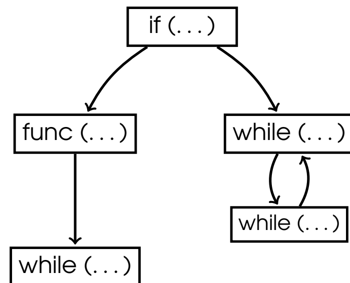

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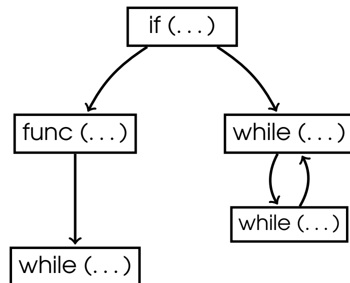

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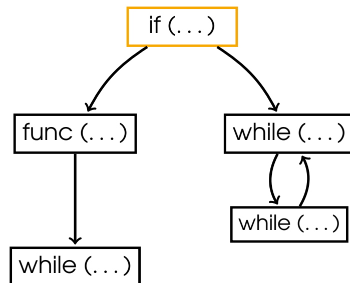

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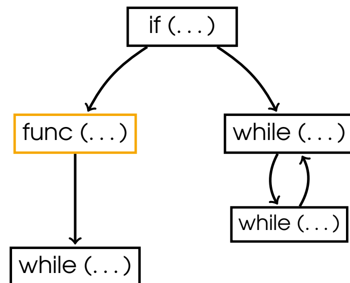

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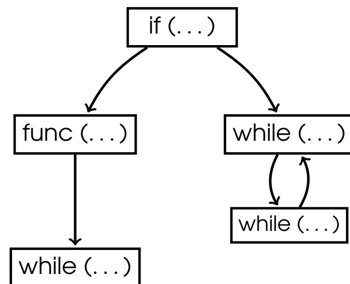

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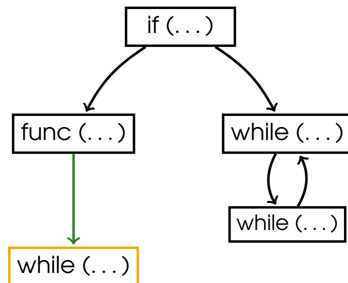

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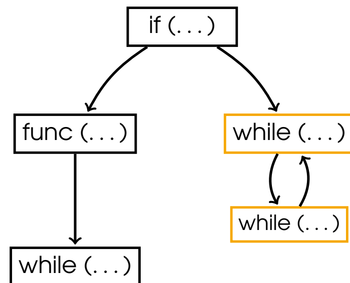

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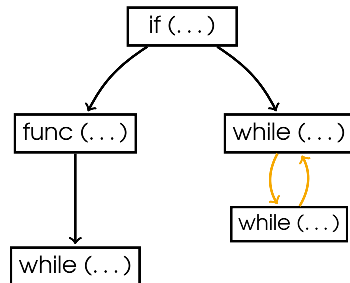

Automatically Computing Runtime Bounds

Step 1: Analyzing the first loop in the main-function

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runtime bound of first loop · size(a)

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$$x \cdot \text{size}(a)$$

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Step 3: Analyzing the size of y in the main-function

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    - Value after n loop iterations:

$$size(y) + n \cdot size(fact(x))$$

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Automatically Computing Size Bounds

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$1 \cdot \log(\text{size}(y))$

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Live Demo

```
(GOAL COMPLEXITY)
(STARTTERM (FUNCTIONSYMBOLS l0))
(RETURNLOCATIONS (FUNCTIONSYMBOLS f2))
(VAR x y a)
(RULES
  l0(x,y,a) -> l1(x,y,a)
  l1(x,y,a) -> l1(x-1,y+f1[a|(x,y,a)<-(x,y,x)],a) :|: x > 0
  l1(x,y,a) -> l2(1,y,a) :|: x <= 0
  l2(x,y,a) -> l2(3*x,2*y,a) :|: x < y && x > 0

  f1(x,y,a) -> f2(x,y,1) :|: a = 0
  f1(x,y,a) -> f2(x,y,a*f1[a|(x,y,a)<-(x,y,a - 1)]) :|: a > 0)
```

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	$\mathcal{O}(1)$	$\mathcal{O}(\log(n))$	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^{>2})$	$< \infty$	$\mathrm{AVG}^+(\mathrm{s})$	$\mathrm{AVG}(\mathrm{s})$
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KoAT1	132	0	231	108	16	499	0.64	8.06
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CoFloCo	125	0	234	95	10	464	3.19	16.38	71%
KoAT1	132	0	231	108	16	499	0.64	8.06	76%
KoAT2	128	11	281	121	28	583	3.45	21.78	89%

- At most 652 benchmarks might terminate
- KoAT2 solves 89% of benchmarks which might terminate

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Thank You!